Calculus Honors | Name: (Guided Notes) And KEY

Derivatives of Inverse Trigonometric Functions | Period: ________ | Date: ______________

For starters, what is an **inverse function**?

Every function has an inverse relation, but that inverse is NOT necessarily a function!

Inverses reverse the ordered pairs: the inputs of a function become the outputs of its inverse, and the outputs of the function become the inputs of its inverse.

A function's inverse is also a function IF the original function passes BOTH the horizontal and vertical line tests. That is, if the function is a one-to-one function then its inverse is also a function.

Notation: Inverse functions are denoted by what LOOKS like an exponent of -1. This does NOT mean "raise the function to the power of negative one". It simply stands for the inverse function.

That is, $f^{-1}(x) \neq \frac{1}{f(x)}$!!!!!!

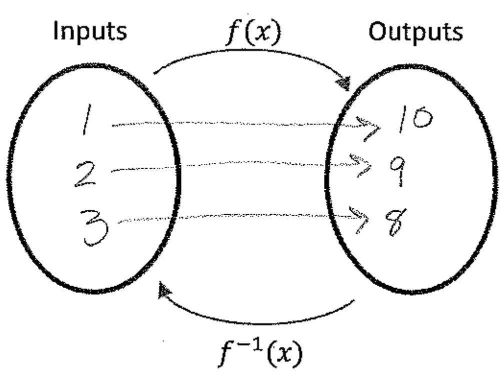
Inverse functions "undo" each other. So, composition of inverse functions effectively cancel each other out:

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x$$

Here are three ways to represent, or think about, inverse functions.

1) Mapping Diagram

Mapping diagrams show the scheme of mapping inputs to outputs by a function.



$f(1) = 10$ $\quad f^{-1}(10) = 1$

$f(2) = 9$ $\quad f^{-1}(9) = 2$

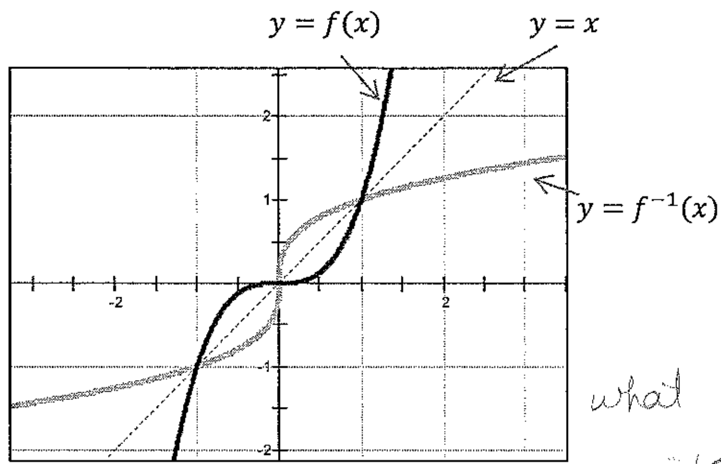
$f(3) = 8$ $\quad f^{-1}(8) = 3$

2) Graph

Graphs of inverse functions are reflections over the line $y = x$.

Do you recognize this function?

$f(x) = x^3$



all ordered pairs in $f(x)$ are reversed in $f^{-1}(x)$

what is $f^{-1}(x)$?

$f^{-1}(x) = \sqrt[3]{x}$

3) Algebra

To find an inverse equation, write the function as an equation in y and x.
Then, change all the x's to y's and all the y's to x's.
This is the inverse equation!
Solve this equation for y and graph it. If this passes the vertical line test, then it is a function, and it is the inverse function of the original function you started with.

$y = x^3$
$x = y^3$
$y = \sqrt[3]{x}$

Now, back to **inverse trig** functions.

We can't use the algebra technique to find the inverse functions, because we don't actually have an algebraic expression for the trig functions. All we have is $y = \sin(x)$. So, we can say $x = \sin(y)$ is the inverse equation, but how do we solve that for y?

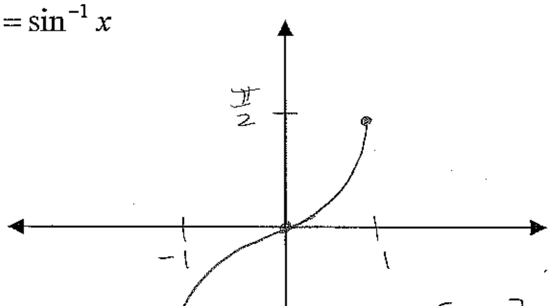
We use the inverse notation: $y = \sin^{-1}(x)$, or we use the alternate notation for trig inverses, which is to put the word arc in front of the trig function: $y = arc\sin(x)$. These both mean, the inverse of the sine function.

$y = \sin^{-1} x \equiv y = \arcsin x$ $\quad \csc^{-1} x \equiv \text{arccsc}\, x$
$\cos^{-1} x \equiv \arccos x$ $\quad \sec^{-1} x \equiv \text{arcsec}\, x$
$\tan^{-1} x \equiv \arctan x$ $\quad \cot^{-1} x \equiv \text{arccot}\, x$

So, how do we figure out what that looks like, and how to find its derivative?

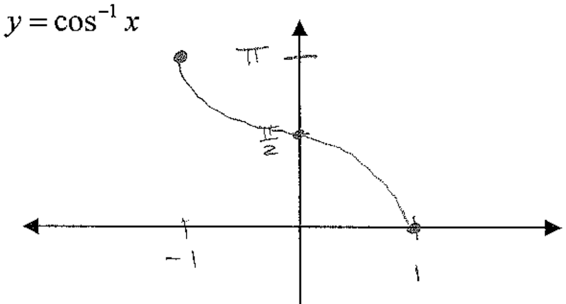
Draw a sketch of each inverse function and identify the domain and range.

$y = \sin^{-1} x$



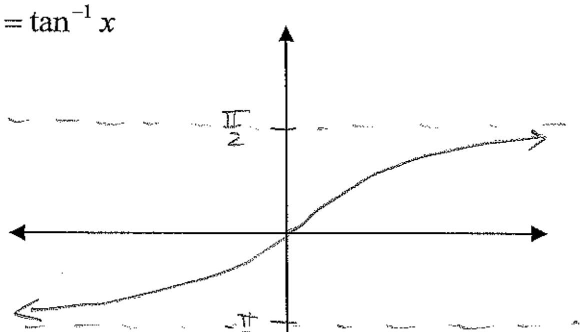
D: $x \in [-1, 1]$
R: $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = \cos^{-1} x$



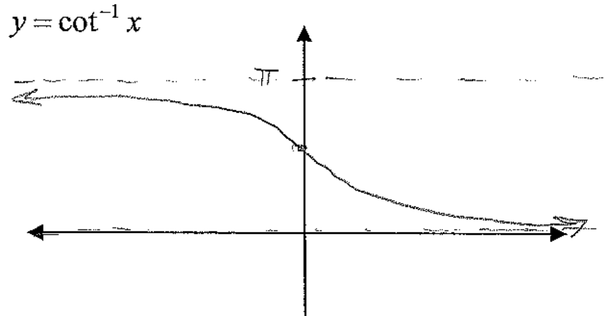
D: $x \in [-1, 1]$
R: $y \in [0, \pi]$

$y = \tan^{-1} x$



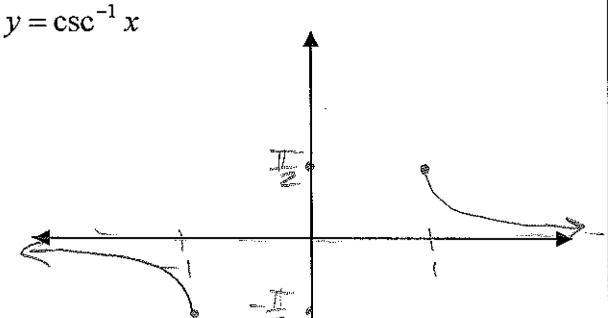
D: $x \in \mathbb{R}$
R: $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$y = \cot^{-1} x$



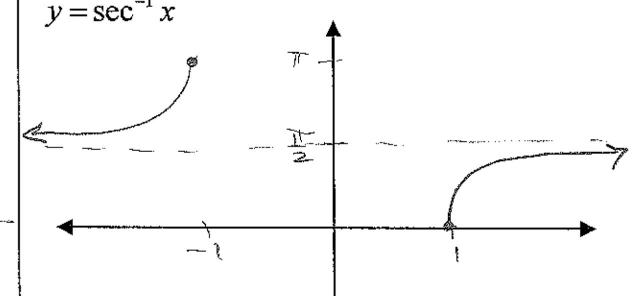
D: $x \in \mathbb{R}$
R: $y \in (0, \pi)$

$y = \csc^{-1} x$



D: $x \in (-\infty, -1] \cup [1, \infty)$
R: $y \in [-\frac{\pi}{2}, \frac{\pi}{2}], y \neq 0$

$y = \sec^{-1} x$



D: $x \in (-\infty, -1] \cup [1, \infty)$
R: $y \in [0, \pi], y \neq \frac{\pi}{2}$

Notes about Inverses:

- ✓ inverse functions reverse ordered pairs
- ✓ Inverse functions compose to x
- ✓ Not all inverses are functions – might need restrictions
- ✓ Inverse notation uses $f^{-1}(x)$ → does NOT mean $\frac{1}{f(x)}$!
- ✓ Graphs of inverse functions reflect over the line $y=x$

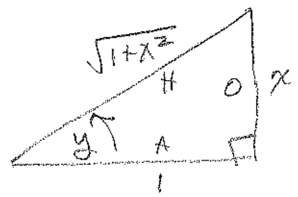
Finding Derivatives of Inverse Trig Functions

If $y = tan^{-1}(x)$, then $y' = ?$

Step 1: Rewrite without inverses -

$$\tan y = x$$

Step 2: Draw a representative triangle



$$\tan y = \frac{O}{A} = \frac{x}{1}$$

$$1^2 + x^2 = c^2$$

$$c = \pm\sqrt{1+x^2}$$

Step 3: Take $\frac{d}{dx}$ of both sides (use chain rule!)

$$\frac{d}{dx}[\tan y] = \frac{d}{dx}[x]$$

$$\sec^2 y \cdot y' = 1$$

Step 4: Solve for y'

$$y' = \frac{1}{\sec^2 y}$$

Step 5: Use triangle to rewrite in terms of x.

$$\sec y = \frac{H}{A} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$$

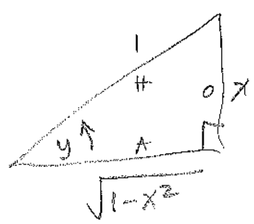
$$y' = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$$

Finding Derivatives of Inverse Trig Functions

If $y = sin^{-1}(x)$, then $y' = ?$

Step 1: $\sin y = x$

Step 2:



$\sin y = \frac{O}{H} = \frac{x}{1}$

Step 3:

$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$

$\cos y \cdot y' = 1$

Step 4:

$y' = \frac{1}{\cos y}$

Step 5:

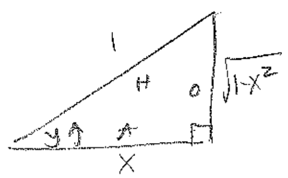
$\cos y = \frac{A}{H} = \frac{\sqrt{1-x^2}}{1}$ $\qquad y' = \frac{1}{\sqrt{1-x^2}}$

Finding Derivatives of Inverse Trig Functions

If $y = cos^{-1}(x)$, then $y' =?$

Step 1: $\cos y = x$

Step 2:



$\cos y = \frac{A}{H} = \frac{x}{1}$

Step 3:

$\frac{d}{dx}[\cos y] = \frac{d}{dx}[x]$

$-\sin y \cdot y' = 1$

Step 4:

$y' = -\frac{1}{\sin y}$

Step 5:

$\sin y = \frac{O}{H} = \frac{\sqrt{1-x^2}}{1}$

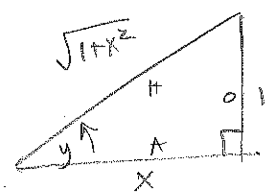
$y' = \frac{-1}{\sqrt{1-x^2}}$

Finding Derivatives of Inverse Trig Functions

If $y = cot^{-1}(x)$, then $y' = ?$

Step 1: $\cot y = x$

Step 2:



$\cot y = \frac{A}{O} = \frac{x}{1}$

Step 3:

$\frac{d}{dx}[\cot y] = \frac{d}{dx}[x]$

$-\csc^2 y \cdot y' = 1$

Step 4:

$y' = \frac{-1}{\csc^2 y}$

Step 5:

$\csc y = \frac{H}{O} = \frac{\sqrt{1+x^2}}{1}$

$\csc^2 y = (\sqrt{1+x^2})^2 = 1+x^2$

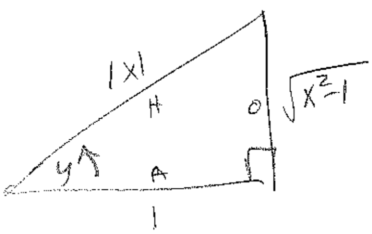
$y' = \frac{-1}{1+x^2}$

Finding Derivatives of Inverse Trig Functions

If $y = sec^{-1}(x)$, then $y' =?$

Step 1: $\sec y = x$

Step 2:



$\sec y = \frac{H}{A} = \frac{x}{1}$

Step 3:

$$\frac{d}{dx}[\sec y] = \frac{d}{dx}[x]$$

$$\sec y \tan y \; y' = 1$$

Step 4:

$$y' = \frac{1}{\sec y \tan y}$$

Step 5:

$\sec y = \frac{H}{A} = \frac{|x|}{1} = |x|$

$\tan y = \frac{O}{A} = \frac{\sqrt{x^2-1}}{1}$

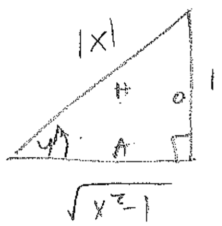
$$y' = \frac{1}{|x|\sqrt{x^2-1}}$$

Finding Derivatives of Inverse Trig Functions

If $y = csc^{-1}(x)$, then $y' = ?$

Step 1: $\csc y = x$

Step 2:



$\csc y = \frac{H}{O} = \frac{x}{1}$

Step 3:

$\frac{d}{dx}[\csc y] = \frac{d}{dx}[x]$

$-\csc y \cot y \cdot y' = 1$

Step 4:

$y' = \frac{-1}{\csc y \cot y}$

Step 5:

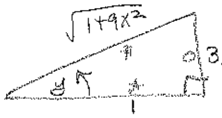
$\csc y = \frac{H}{O} = \frac{|x|}{1}$

$\cot y = \frac{A}{O} = \frac{\sqrt{x^2-1}}{1}$

$y' = \frac{-1}{|x|\sqrt{x^2-1}}$

Practice Problems: Find the derivative of each function, using the triangle technique.

1. $y = tan^{-1}(3x)$ $\quad \tan y = 3x$



$\tan y = \frac{O}{A} = \frac{3x}{1}$

$\frac{d}{dx}[\tan y] = \frac{d}{dx}[3x]$

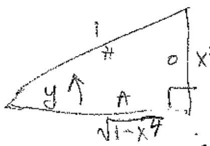
$\sec^2 y \cdot y' = 3$

$y' = \frac{3}{\sec^2 y}$

$\sec y = \frac{H}{A} = \frac{\sqrt{1+9x^2}}{1}$

$\boxed{y' = \frac{3}{1+9x^2}}$

2. $y = sin^{-1}(x^2)$ $\quad \sin y = x^2$



$\sin y = \frac{O}{H} = \frac{x^2}{1}$

$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x^2]$

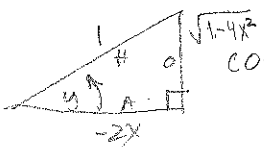
$\cos y \cdot y' = 2x$

$y' = \frac{2x}{\cos y}$

$\cos y = \frac{A}{H} = \frac{\sqrt{1-x^4}}{1}$

$\boxed{y' = \frac{2x}{\sqrt{1-x^4}}}$

3. $y = cos^{-1}(-2x)$ $\quad \cos y = -2x$



$\cos y = \frac{A}{H} = \frac{-2x}{1}$

$\frac{d}{dx}[\cos y] = \frac{d}{dx}[-2x]$

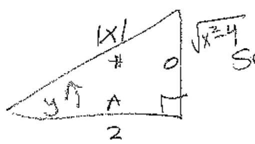
$-\sin y \cdot y' = -2$

$y' = \frac{2}{\sin y}$

$\sin y = \frac{O}{H} = \frac{\sqrt{1-4x^2}}{1}$

$\boxed{y' = \frac{2}{\sqrt{1-4x^2}}}$

4. $y = sec^{-1}\left(\frac{x}{2}\right)$ $\quad \sec y = \frac{x}{2}$



$\sec y = \frac{H}{A} = \frac{x}{2}$

$\frac{d}{dx}[\sec y] = \frac{d}{dx}\left[\frac{x}{2}\right]$

$\sec y \tan y \; y' = \frac{1}{2}$

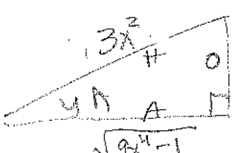
$y' = \frac{1}{2\sec y \tan y}$

$\sec y = \frac{H}{A} = \frac{|x|}{2}$

$\tan y = \frac{O}{A} = \frac{\sqrt{x^2-4}}{2}$

$y' = \frac{1}{\cancel{2}\frac{|x|}{\cancel{2}} \cdot \frac{\sqrt{x^2-4}}{2}} \Rightarrow \boxed{y' = \frac{2}{|x|\sqrt{x^2-4}}}$

5. $y = csc^{-1}(3x^2)$ $\quad \csc y = 3x^2$



$\csc y = \frac{H}{O} = \frac{3x^2}{1}$

$\frac{d}{dx}[\csc y] = \frac{d}{dx}[3x^2]$

$-\csc y \cot y \, y' = 6x$

$y' = \frac{6x}{-\csc y \cot y}$

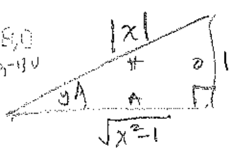
$\csc y = \frac{H}{O} = \frac{3x^2}{1}$

$\cot y = \frac{A}{O} = \frac{\sqrt{9x^4-1}}{1}$

$y' = \frac{6x}{-3x^2\sqrt{9x^4-1}} \Rightarrow \boxed{y' = -\frac{2}{x\sqrt{9x^4-1}}}$

6. $y = sin^{-1}\left(\frac{1}{x}\right)$ $\quad \sin y = \frac{1}{x}$

let $u = \frac{1}{x}$
$\sin^{-1} u \Rightarrow D: u \in [-1,1]$
$x = \frac{1}{u} \Rightarrow x \in (-\infty,-1] \cup [1,\infty)$



$\sin y = \frac{O}{H} = \frac{1}{x}$

$\frac{d}{dx}[\sin y] = \frac{d}{dx}\left[\frac{1}{x}\right]$

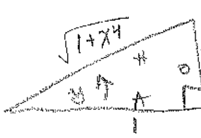
$\cos y \cdot y' = -\frac{1}{x^2}$

$y' = -\frac{1}{x^2 \cos y}$

$\cos y = \frac{A}{H} = \frac{\sqrt{x^2-1}}{|x|}$

$y' = -\frac{1}{x^2 \cdot \frac{\sqrt{x^2-1}}{|x|}} \Rightarrow \boxed{y' = -\frac{1}{|x|\sqrt{x^2-1}}}$

7. $y = tan^{-1}(-x^2)$ $\quad \tan y = -x^2$



$\tan y = \frac{O}{A} = \frac{-x^2}{1}$

$\frac{d}{dx}[\tan y] = \frac{d}{dx}[-x^2]$

$\sec^2 y \cdot y' = -2x$

$y' = \frac{-2x}{\sec^2 y}$

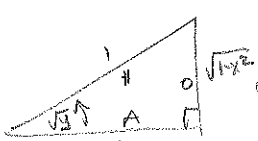
$\sec y = \frac{H}{A} = \frac{\sqrt{1+x^4}}{1}$

$\sec^2 y = 1 + x^4$

$\boxed{y' = \frac{-2x}{1+x^4}}$

8. $y = (cos^{-1}(x))^2$ $\quad \sqrt{y} = \cos^{-1} x$

$\cos\sqrt{y} = x$



$\cos\sqrt{y} = \frac{A}{H} = \frac{x}{1}$

$\frac{d}{dx}[\cos\sqrt{y}] = \frac{d}{dx}[x]$

$-\sin\sqrt{y} \cdot \frac{1}{2}y^{-\frac{1}{2}} \cdot y' = 1$

$-\sin\sqrt{y} \cdot \frac{1}{2\sqrt{y}} \cdot y' = 1$

$y' = \frac{-2\sqrt{y}}{\sin\sqrt{y}}$

$\sin\sqrt{y} = \frac{O}{H} = \frac{\sqrt{1-x^2}}{1}$

$\sqrt{y} = \cos^{-1} x$

$\boxed{y' = \frac{-2\cos^{-1} x}{\sqrt{1-x^2}}}$