Series

A **series** is a sum of the terms in a sequence.

There are many useful applications of mathematical series.

Examples:

- A theater has 14 rows of seats. The first row has 20 seats, and each row after that has two more seats than the row before it. How many seats does the theater hold?

- A supermarket manager wants to create a pyramid display of cans of yams. If the bottom layer has 31 cans of yams, and each layer above it has two fewer cans than the layer below, how many layers will the cans stack until there is only one can? How many cans total are there in the pyramid?

- A tennis ball is dropped from a 15 foot platform and bounces. Each time it bounces up, it reaches half of its previous height. How far (vertically) in total does the ball bounce before it comes to rest?

There are many more situations for which sequences and series are excellent mathematical models. Luckily, we have some straight-forward methods for solving these problems. ☺

Related Series

Each sequence has a related series. The sequence is the list of terms. The series is the sum of all those terms.

Example:

a) Find the related series given the sequence 1, 3, 5, 7, 9, …
Solution: The related series is the sum of all the terms: 1+3+5+7+9+…

In this example, there is NO sum, because we would keep adding forever.

b) Find the related series given the partial sequence 1, 5, 9, 13, 17, 21
Solution: The related series is the sum of the terms: 1+5+9+13+17+21=66
Because this is the sum of the first 6 terms of a sequence, we call it the 6^{th} partial sum.

Notation

The symbol for a series, or sum, is S.

The symbol for a partial sum is S_n, where n is the number of terms you are adding.

Formula

When the sequence is arithmetic, we have NO SUM for the infinite series.

However, we have a nice formula for the n^{th} partial sum, S_n.

Story time:

Once upon a time, in the late 1700's, there was a brilliant young student named Carl Friedrich Gauss. The story goes that he was so brilliant that his teachers had trouble keeping him busy while the other students learned what came so easily to him. One day, a teacher asked him to add all the numbers from 1 to 100, just to keep him occupied for a while. (This was WAY before calculators!) In a moment or so, he gave the correct answer. The teacher was astonished. She asked him how he calculated that so quickly.

So, what is the correct answer, and how did he do it?

Consider:

1 + 2 + 3 + 4 + 5 + . . . + 96 + 97 + 98 + 99 + 100

Because with addition, order doesn't matter, young Gauss regrouped the sum in pairs, working outside in. What do you notice about these pairs?

1 + 100 =

2 + 99 =

3 + 98 =

Will this continue? Certainly! How many pairs will add up to 101? Then, what is the sum?

Using this method, we have the formula for any finite arithmetic series,

$$S_n = \frac{n}{2}(a_1 + a_n)$$

where n is the number of terms to add up, a_1 is the first term in the sum and a_n is the last term in the sum. Note that this formula ONLY works for arithmetic series!

<u>More Notation</u>

Writing out a sum can be tedious, and sometimes there are too many terms to write them all out.

For example, how would we write the series related to the first 100 terms of the arithmetic sequence, $a_n = 2n - 1$?

1 + 3 + 5 + 7 + 9 + 11 + … + 199

There is another way to write this, using **sigma notation**.

The Greek letter that corresponds to our letter S is sigma. Greek capital sigma looks like this: Σ

Maybe you have seen this symbol when using an excel document? It means, "add everything up."

That is what the sigma notation means: add up all the terms indicated by the notation.

We can write the sigma, then the formula for the sequence.

$$\sum(2n-1)$$

But, that doesn't give us enough information. We need to know how many terms to add up!

So, we have a **summation index** on the sigma, to indicate which term to start with, and which term to end with.

$$\sum_{n=1}^{100}(2n-1)$$

The index, in this case, is n. The $n = 1$ under the sigma means, "start with the first term". The 100 above the sigma means "End with the 100^{th} term."

So, the sigma notation is shorthand for our expanded sum:

$$\sum_{n=1}^{100}(2n-1) = 1 + 3 + 5 + 7 + \cdots + 199$$

Now we can use the formula to find the sum:

$$S_{100} = \frac{100}{2}(1 + 199) = 50(200) = 10000$$

Note: We can make the summation index any letter we want, but then we have to use that same letter in the formula in place of n.

Skills

❶ Evaluate a related series.

Technique: Add up the terms!

Example:

Evaluate the series related to the given partial sequence: $-10, -5, 0, 5, 10, 15, 20$

Solution: $-10 + \ -5 + 0 + 5 + 10 + 15 + 20 = 35$

❷ Rewrite a series given in sigma notation as a sum. (This is expanding the sum.)

Technique: Using the index, substitute for the terms required one at a time, and list the terms as a sum.

Example:

Rewrite the series as a sum:

$$\sum_{n=1}^{5}(3n-1)$$

Solution:

$$(3(1)-1)+(3(2)-1)+(3(3)-1)+(3(4)-1)+(3(5)-1) = 2+5+8+11+14$$

Note that rewriting it as a sum does NOT ask you to evaluate. This sum *evaluates* to 40, but that is not what is being asked here. The expansion of the terms above is the final answer for this problem.

❸ Evaluate an arithmetic series given sigma notation.

Technique: Determine the number of terms, the first term and the last term. Then use the formula for a finite arithmetic series.

Example:

Evaluate the arithmetic series described:

$$\sum_{n=1}^{20}(4n+3)$$

Solution:

We want to use the formula

$$S_n = \frac{n}{2}(a_1 + a_n)$$

First find the number of terms. To find the number of terms in the sum, subtract the starting number from the ending number and then add 1. $20 - 1 + 1 = 20$. There are 20 terms in this sum, so in our formula, $n = 20$.

Next, we need the first term, a_1. The first term in this series is $4(1) + 3 = 7$. So, $a_1 = 7$.

Next, we need the last term, a_{20}. The last term in this series is $4(20) + 3 = 83$. So, $a_{20} = 83$.

Now, we can substitute:

$$S_{20} = \frac{20}{2}(7 + 83) = 10(90) = 900$$

❹ Determine the number of terms in an arithmetic series given the sum.

Technique: Set up an equation using the formula for a finite arithmetic series and then solve for n.

Example:

Determine the number of terms n in the series described.

$$\sum_{k=1}^{n}(k - 5) = 45$$

Solution:

We are given that $S_n = 45$. Find expressions for the first term and the nth term.

$a_1 = 1 - 5 = -4$ and $a_n = n - 5$

Now create the equation using the formula:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$45 = \frac{n}{2}(-4 + n - 5)$$

Solve for n:

$$90 = n(-9 + n)$$
$$90 = -9n + n^2$$
$$n^2 - 9n - 90 = 0$$

$$(n + 6)(n - 15) = 0$$
$$n = -6 \quad or \quad n = 15$$

Since we can't have a negative number of terms, the only solution is 15. There are 15 terms in the sum.

❺ Use a graphing calculator to evaluate a series.

Technique: Enter a sum using sigma notation in your calculator.

There is a template for entering these formulas. Make sure the entire sequence formula is in parentheses after the sigma symbol.

✎ Now, practice the odd-numbered problems on the 9-4 practice sheet.