Name: ________________________________ Date: ________________________________
Period: ______ Calculus Honors: 4-2 The Product Rule

Warm Up:

1. Factor and simplify.

$$30(3x-5)^9(5x-3)^2+10(3x-5)^{10}(5x-3)$$

2. Find $f'(x)$ if $f(x)=x^2(2x^3+3)$

How did you go about finding the derivative?

Let's Explore how to differentiate the product of two functions!

1. Let $g(x) = x^7$ and let $h(x) = x^{11}$.
   a. Find $g'(x)$.

   b. Find $h'(x)$.

2. Let $f(x) = g(x) \bullet h(x)$.Write an equation for $f(x)$ as a single power of x.

3. Find $f'(x)$.

4. True or False? $f'(x) = g'(x) \bullet h'(x)$ Show work to support your answer.

5. BE CLEVER!!! Use $g(x)$, $g'(x)$, $h(x)$, and $h'(x)$ to find the correct answer for $f'(x)$.

6. Make a conjecture about what $f'(x)$ equals in terms of $g(x)$, $h(x)$, $g'(x)$ and $h'(x)$.

7. TEST YOUR CONJECTURE!!! If $f(x) = x^2 \sin x$, find $f'(x)$. Check on your calculator!

The Product Rule:

If $y = u \cdot v$, then $y' =$ *or* $y' =$

Now, try the Warm Up using the product rule!

Find $f'(x)$ if $f(x) = x^2(2x^3 + 3)$

Now, let's practice!

Ex. 1. $y = 5x\sin x$

Ex. 2. $y = x^2 \ln(\cos x)$

Ex. 3. $y = (3x-5)^{10}(5x-3)^2$

Ex. 4. $y = x^2 \sin x \ln x$

Name______________________________ Product Rule Worksheet Period____

Calculate the derivatives of each of the following.

1) $y=\left(x^2+3\right)\left(x^3-3x+1\right)$ ______________________

2) $y=\left(x^3-2x^2+5\right)\left(x^4-3x^2+2\right)$ ______________________

3) $y=(3x+4)\left(x^3-2x+1\right)$ ______________________

4) $y=\left(x^{3/2}-4x\right)\left(x^4-\frac{3}{x^2}+2\right)$ ______________________

5) $y=\left(x^2-4\right)\left(x^4+2x+1\right)$ ______________________

6) $y=\left(\sqrt{x}+3x\right)\left(5x^2-\frac{3}{x}\right)$ ______________________

7) $y=\sin(2x)\cos(2x)$ ______________________

8) $y=3^{x^3+2x+1}\cos x$ ______________________

9) $y=5\sin\left(x^2\right)\cdot e^{5x}$ ______________________

10) $y=\left(3x^2+5x-6\right)\ln(3x+5)$ ______________________

11) $y=(5x+3)\ln\left(\sqrt{x+2}\right)$ ______________________

12) $y=x^{2/3}\left(x^2-2\right)\left(x^3-x+1\right)$ ______________________

4-3 The Quotient Rule

Warm Up:

Find $f'(x)$ if

$$f(x)=\frac{x^2}{x}$$

Conclusion: The derivative of a quotient is NOT simply the quotient of the derivatives

The Quotient Rule:

If $y=\frac{u}{v}$, then $y'=\frac{vu'-uv'}{v^2}$

Now, try the Warm Up using the product rule!

Find $f'(x)$ if

$$f(x)=\frac{x^2}{x}$$

Ex. 1. $y=\frac{\sin x}{x^2}$

Ex. 2. $y=\frac{x-1}{x+2}$

Ex. 3.

Find the equation of the tangent line to $f(x)=\frac{1+x}{1-x}$ at $x=2$.

Quotient Rule Worksheet
Calculate the derivatives of each of the following.

Part 1: Use the quotient rule to find the derivative of each of the following.

1) $f(x)=\dfrac{6x-8}{12-8x}$

2) $f(x)=\dfrac{7x-2\sqrt{x}}{3x^2-2}$

3) $f(x)=\dfrac{\cos x}{x^2}$

4) $f(x)=\dfrac{e^{x^2}}{\tan 2x}$

5) $y=\cot(5x)$

6) $y=\sec x\cot x$

7) $y=\sec^3\left(\dfrac{x}{4}\right)$

8) $y=\ln(\cos x)$

Part 2: Find the second derivative of each of the following.

1) $y=\ln\left(x^2+5x+6\right)$

2) $y=\ln(\sin x)$

3) $y=\ln\left(x^3+3x-4\right)$

4) $y=\ln\left(2^x\right)$

Part 3: Write each of the following in terms of $\sin x$ and/or $\cos x$.

1) $\tan x$ ____________

2) $\cot x$ ____________

3) $\sec x$ ____________

4) $\csc x$ ____________

Part 4: Use the product and/or quotient rule to find the derivatives of the trigonometric functions above.

1) $\frac{d}{dx}\tan x$ __________________

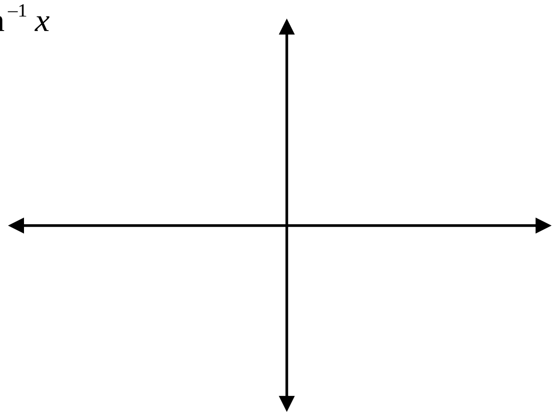
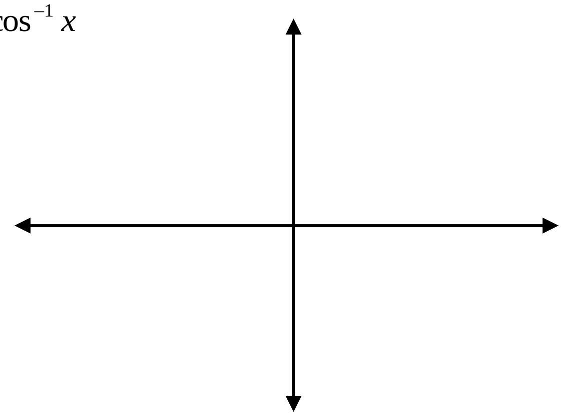
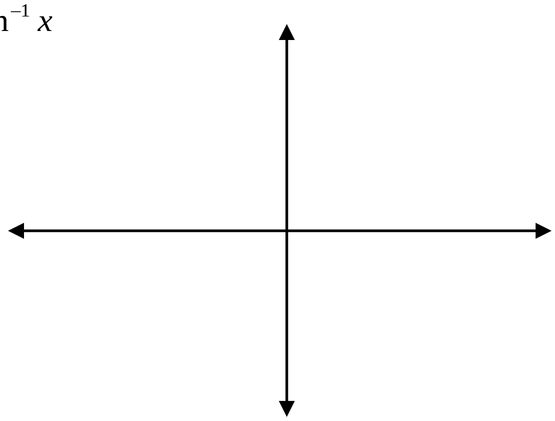
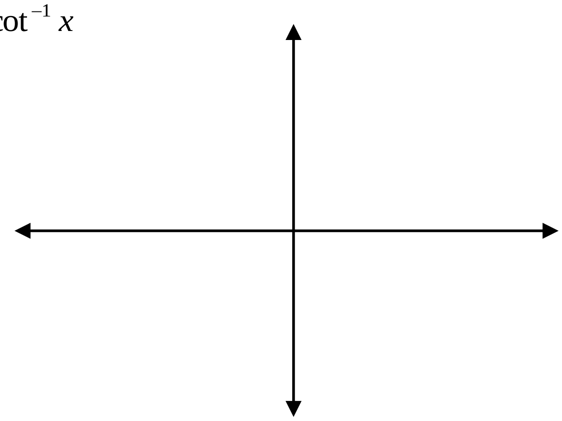
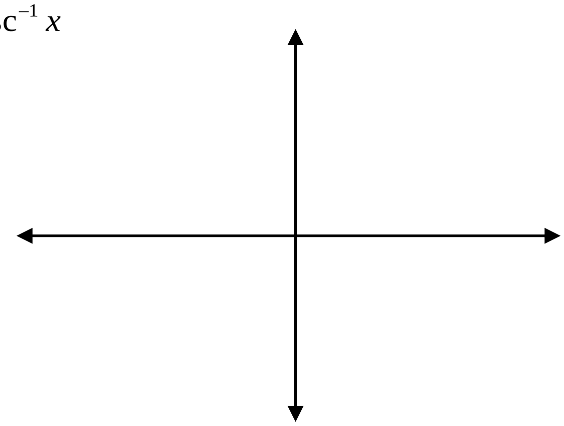
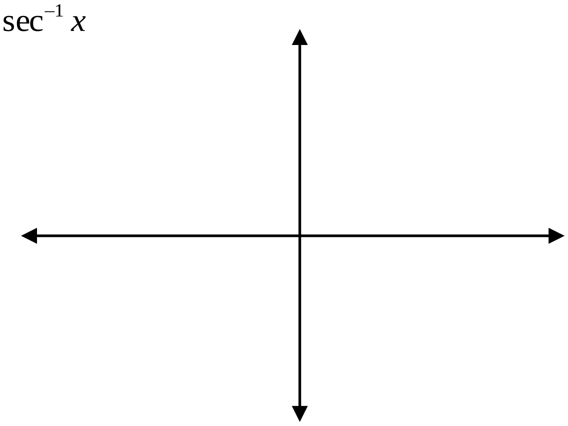
2) $\frac{d}{dx}\cot x$ __________________

3) $\frac{d}{dx}\sec x$ __________________

4) $\frac{d}{dx}\csc x$ __________________

Inverse Trig Functions Review

Draw a sketch of each inverse function and identify the range.

$y = \sin^{-1} x$ 	$y = \cos^{-1} x$ 
$y = \tan^{-1} x$ 	$y = \cot^{-1} x$ 
$y = \csc^{-1} x$ 	$y = \sec^{-1} x$ 

Notes about Inverses:

- ✓
- ✓
- ✓
- ✓
- ✓

4.6 Differentiability and Continuity

If a function f has a value for $f'(c)$, then f is said to be **differentiable** at $x = c$.

If f is differentiable at every value of x in an interval, then f is said to be differentiable on that interval.

Definitions:

- **Differentiability at a point**: Function f is **differentiable at** $x = c$ if and only if $f'(c)$ exists. (That is, $f'(c)$ is a real number.)
- **Differentiability on an interval:** Function f is **differentiable on an interval** if and only if it is differentiable for every *x*-value in the interval.
- **Differentiability:** Function f is **differentiable** if and only if it is differentiable at every value of *x* in its domain.

If a function is defined on a closed interval, then it can only be differentiable on the open-interval because taking the limit at a point requires being able to approach the point from both sides.

Property: Differentiability Implies Continuity
If a function f is differentiable at $x = c$, then f is continuous at $x = c$.

If function f is not continuous at $x = c$, then f is not differentiable at $x = c$. This is the CONTRAPOSITIVE of the property above. (If there's a hole in a graph it won't work out.)

Looking at graphs is a good way to determine differentiability (and continuity.)

Example 1
Prove that $f(x) = x^2 - 8x + 3$ is continuous at x = 3.

Example 2
Is the function $f(x) = \dfrac{(x-4)(x+5)}{(x-4)}$ differentiable at x = 4? Justify your answer.

Example 3

Let $f\left(x\right)=\begin{cases} ax^3 & x\le 2 \\ b(x-3)^2+10 & x>2 \end{cases}$. Find the values of a and b such that $f\left(x\right)$ is differentiable at $x=2$.

Example 4

Let $f\left(x\right)=\begin{cases} bx^2+6x & x\le 2 \\ ax^3 & x>2 \end{cases}$. Find the values of a and b such that $f\left(x\right)$ is differentiable at $x=2$.

4.2- 4.6 Review

Simplify answers completely. Show all work.

1-8 Differentiate.

1) $f(x) = \dfrac{4^x}{\tan 7x}$ ________________________

2) $y = \cot\left(x^3\right)$ ________________________

3) $y = \sin x \cot x$ ________________________

4) $y = \csc^4\left(\sqrt[3]{x}\right)$ ________________________

5) $y = \ln\left(\sqrt{x+3}\right)$ ________________________

6) $f(x) = \dfrac{3x-8}{2-11x}$ ________________________

7) $f(x) = e^x \cdot (4x-5)^3$ ________________________

8) $f(x) = \dfrac{\sqrt{x}}{\cos x}$ ________________________

9. Find the derivative using implicit differentiation. $y = \tan^{-1}(3x^7)$. ________________________

Make sure to show your triangles.

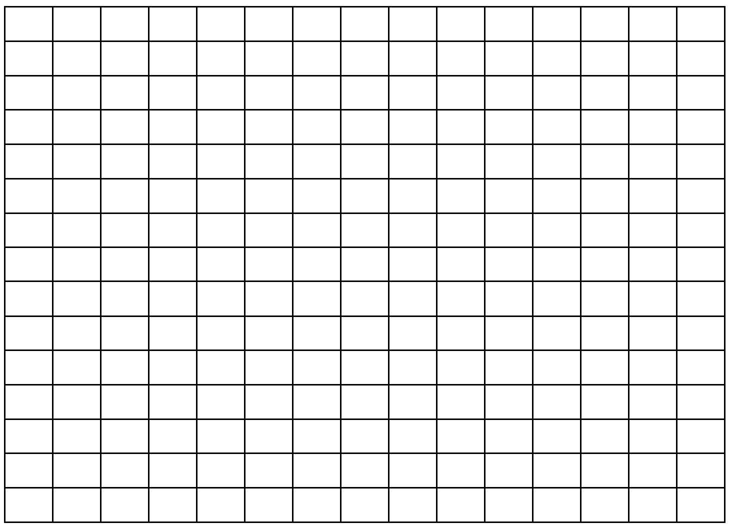
10. Let $f(x) = \begin{cases} ax^3 + 1, & x \le 1 \\ (x-2)^2 + b, & x > 1 \end{cases}$.

Show all work and use proper limit notation.

Find the values of *a* and *b* such that $f(x)$ is differentiable at x = 1. a = __________ b= __________

4.7 Derivatives of Parametric Equations/Curves

WARMUP:
Use a graphing calculator to sketch the graph of the following curve on [-3,3].
$x = t^2$ and $y = t - 1$



Parametric equations: $(x(t), y(t))$ where t is over some specified domain.

Curve is all points on the graph over the indicted *t* interval.

Typically parametric equations are sets of ordered pairs that mark off a path that a particle follows over time, so think:

It's at $(0,0)$ when $t = 0$
It's at $(1,4)$ when $t = 1$
It's at $(0,5)$ when $t = 2$

This is a lame example and we don't get all that much information from it.

As the change in t gets smaller and smaller the picture gets better and better.

Creating Parametric Graphs:

- We can create graphs of parametric equations by point plotting (in a bad scenario)
- We can create graphs of parametric equations by eliminating the parameter (but you loose the orientation of the curve)
- We can create a very useful graph of the parametric equations using a calculator (you can trace and watch it move)

What can derivatives tell us about parametric equations?

We can find lots of derivatives (YAY!)

- $\frac{dx}{dt}$ tells us how the particle is moving from left to right (it cares nothing about up and down)
- $\frac{dy}{dt}$ tells us how the particle is moving up and down (tells us nothing about left and right)
- $\frac{dy}{dx}$ tells us the relationship between the rate of change of *y* and the rate of change of *x*…which is exactly the same thing that it's always told us, slopes of tangent lines

How to find $\frac{dy}{dx}$:

- We need both x and y to be differentiable functions of t, then the slope of the *xy*-graph is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
- If $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$ then the graph has a horizontal tangent
- If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ then the graph has a vertical tangent

Example 1 of things to do:

$$\begin{cases} x(t) = t^2 - 9 \\ y(t) = t^2 - 8t \end{cases}$$

Sketch the curve after graphing it on your calculator.

Find the equation of the tangent line at $t = 4$.

Find the points where the tangent has slope $\frac{1}{2}$.

Find the points where the tangent is horizontal or vertical.

Example 2 of things to do:

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following curves:

$$c_1(t) = (2t+1, 1-9t)$$

$$c_2(t) = \begin{cases} x(t) = \frac{1}{2}t \\ y(t) = \frac{1}{4}t^2 - t \end{cases}$$

$$c_3(s) = \left(s^3, s^6 + s^{-3}\right)$$

$$c_4(\theta) = \left(\cos\theta, \cos\theta + \sin^2\theta\right)$$

4.8 Graphs and Derivatives of Implicit Relations

Explicit functions: can be solved for *y* without "resorting to cases." This is because y is defined explicitly in terms of x.

Ex. $y = 3x + 5$

Implicit functions either cannot be solved for *y* or cannot be solved for *y* without resorting to some cases. This is because the relations are implied by an equation.

Ex. $x^3y^2 - 5xy\left(3x + 7y^5\right) = 8$

Think about how you would graph a circle on your calculator when you're not in parametric mode.

Implicit Form	Explicit Form	Derivative
$x^2y = 2$	$y = \frac{2}{x^2}$	

Sometimes working with implicit functions is so much easier that you wouldn't even bother trying to solve for *y*.

Implicit Differentiation

To find $\frac{dy}{dx}$ *for a relation whose equation is written implicitly:*

1. Differentiate both sides of the equation with respect to x. Obey the chain rule by multiplying by $\frac{dy}{dx}$ each time you differentiate an expression containing y.
2. Isolate $\frac{dy}{dx}$ by getting all of the $\frac{dy}{dx}$ terms onto one side of the equation, and all other terms onto the other side. Then factor, if necessary, and divide both sides by the coefficient of $\frac{dy}{dx}$.

Derivatives of implicit functions really just use the chain rule over and over and over…then you solve for $\frac{dy}{dx}$.

Constantly say this sentence to yourself as you take the derivative: "but y is a function of x so I have to chain rule this thing…"

Example 1:

Find the derivative of $\frac{x^2}{2} + \frac{y^2}{2} = 5$.

Example 2:
Find the derivative of $y^3 + x^2 y^5 - 8x^5 = 24$

Example 3:
Find the derivative of $\sin(x \cdot y) = \frac{1}{2}$

Example 4:
Find the derivative of $x^3 y = 5$

Example 5:
Find the equations of the tangent lines to the curve $x^3 + y^2 = 5$ at $x = -3$

4-9 Related Rates

Think about this!

Suppose that two variables *x* and *y* are functions of another variable *t*, say $x = f(t)$ and $y = g(t)$.

We may interpret the derivatives dx/dt and dy/dt as the rates of change of *x* and *y* with respect to *t*.

Two variables x and y are functions of a variable t and are related by the equation $x^3 - 2y^2 + 5x = 16$.

1. If $\frac{dx}{dt} = 4$ when $x = 2$ and $y = -1$, find the corresponding value of $\frac{dy}{dt}$.

2. A ladder 20 feet long leans against a vertical building. If the bottom of the ladder slides away from the building horizontally at a rate of 2 ft/sec, how fast is the ladder sliding down the building when the top of the ladder is 12 feet above the ground?

3. The radius of a sphere is increasing at a constant rate of 0.5 inch/second.

   a. When the radius of the sphere is 15 inches, at what rate is the volume of the sphere changing?
   b. When the volume and radius of the sphere are changing at the same rate, what is the radius of the sphere?

4. A balloon is being inflated at a rate of $10\pi \frac{\text{ft}^3}{\text{sec}}$. At what rate is the radius increasing when $r = 2$ feet?

5. The edges of a cube are increasing at a rate of 2 cm/s.
   a. How fast is the volume of the cube increasing when each edge is 5 cm long?
   b. How fast is the surface area of the cube changing when each edge is 5 cm?

4.9 Related Rates Classwork

A 6 meter ladder is against a wall. If its bottom is pulled at a constant rate of $\frac{1}{2} m/\sec$, how fast is the ladder top sliding when it reaches:

a. 5 meters up the wall?
b. 3 meters up the wall?
c. 1 meter up the wall?

Givens (rates and information)	
Unknown rate	
Formula	

Diagram:

2. A winch (altitude of 20 feet) reels in a rope at a rate of 2 ft/ sec. How fast is the boat moving when the rope is:
   a. 45 feet?
   b. 30 feet?
   c. 22 feet?
   d. 20.05 feet?

Givens (rates and information)	
Unknown rate	
Formula	

Diagram:

3. The edges of a cube are increasing at a rate of 2 cm/s.
   a. How fast is the volume of the cube increasing when each edge is 5 cm long?
   b. How fast is the surface area of the cube changing when each edge is 5 cm?

Givens (rates and information)	
Unknown rate	
Formula	

Diagram: